As regards the remaining spherical functions, they can be expressed through Legendre polynomials as follows:

$$(3) \qquad \begin{aligned} Y_{m,k} &= (\xi_1 - i\xi_2)^k P_m^{(k)}(\xi_3), \qquad (k > 0), \\ Y_{m,-k} &= (\xi_1 + i\xi_2)^k P_m^{(k)}(\xi_3), \qquad (k > 0). \end{aligned}$$

We are going to see this in Section 11, where we will also show that the representation of SO_3 in the space U_m is isomorphic to the representation Ψ_{2m} constructed in 7.4.

Questions and Exercises

1. Suppose the compact group G acts transitively on a topological space X. Let H be the isotropy subgroup of the point $o \in X$. Show that in every finite-dimensional nonzero G-invariant subspace of continuous functions on X there exists a nonzero H-invariant function.

2. Under the assumptions of the preceding exercise, suppose that $T: G \to \mathrm{GL}(V)$ is a finite-dimensional irreducible representation of G. Prove that if V contains a nonzero H-invariant vector, then T is isomorphic to a representation of G in a space of continuous functions on X. (*Hint:* Establish first the existence of a nonzero H-invariant linear function $f \in V'$; then consider the map that assigns to each vector $v \in V$ the continuous function f_v on X defined by the formula $f_v(go) = f(g^{-1}v)$.)

Work out Exercises 3–6 without resorting to formulas (3).

3. Show that $Y_{m,k}(o) = 0$ if $k \neq 0$.

4. Find explicit expressions for the functions $Y_{m,k}$ for $m = 1, 2$.

5. Show that $Y_{m,m} = (\xi_1 - i\xi_2)^m$.

6. Show that $\overline{U}_m = U_m$. Deduce from this that $\overline{Y}_{m,k} = Y_{m,-k}$ (where the bar denotes complex conjugation).

7. Prove the following formula of Rodrigues:

$$P_m(t) = \frac{d^m}{dt^m}\Big[(t^2 - 1)^m\Big].$$